An Adaptive Sequential Competition Test for Beam Selection in Massive MIMO Systems

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Abstract—To solve the problem of beam selection or capturing the highest possible signal power, we propose a sequential test that can adapt to the SNR operating point and speed up the selection procedure in terms of the number of required observations in comparison to a perfectly tuned fixed length test assuming genie knowledge. The speed up gets higher for lower SNR and becomes of particular interest in massive Multiple Input Multiple Output (MIMO) systems using beamforming, where the number of candidate beams is large and exhaustive search can cause intolerable delay due to limited channel coherence time.

Index Terms—Millimeter wave, Massive MIMO, Beam Selection, Sequential Test, Generalized Likelihood Ratio Test

I. INTRODUCTION

In antenna array systems directional signal transmission and/or reception using beamforming is employed to counter the increasing path loss at higher transmission frequencies [1]. A crucial question at the receiver side, sketched schematically for a single plane wave signal (one angle of arrival, AoA) in Fig 1, is how to choose the best beam capturing the highest amount of receive power (an equivalent question arises for an antenna array at the transmitter side). Typically a set of beams (also denoted as the “beam codebook”) is available that can be steered into different mainlobe directions as also illustrated in Fig 1 by a set of 8 orthogonal beam patterns.

Using noisy observations of training signals arriving from one or several directions and possibly different users, this question can be formulated as a hypothesis testing problem where each hypothesis represents one of the candidate beams together. The performance of any such test can be evaluated in terms of the average captured power.

The conventional approach is to estimate the captured power using an observation sequence of fixed length under each beam and to choose the beam with the highest signal power estimate [2], [3]. To use this method efficiently, i.e. not with a much larger number of observations than necessary to achieve the desired performance level, knowledge of the Probability Density Functions (PDFs) under all hypotheses is required. However, this knowledge is not available to the detector due to varying operation conditions under which this problem needs to be solved.

As a remedy we introduce in this work a sequential (variable length) test [4] based on a Generalized Likelihood Ratio (GLR), which learns on the fly the current statistics of the signals in terms of their amplitudes and noise variances based on the available observations. In this way the decision on the best beam can be made as quickly as possible and adaptively according to the SNR operating point. This becomes particularly interesting when the number of beam patterns from which we can choose, becomes large and/or the signal to interference and noise ratio (SINR) is low. This scenario may occur frequently in massive MIMO and mm-wave communication systems which operate at carrier frequencies of several 10 GHz.

In the following, we study in Sec. II the performance of a binary conventional fixed length test under mismatch in the design parameters. Sec. III presents the binary ‘composite’ test using the GLR, that we employ to discriminate between the presence of an unknown DC level versus its absence in white Gaussian noise (WGN) with unknown variance.

II. BEAM SELECTION USING A FIXED LENGTH TEST

We start by looking at the problem of detecting the presence of a known training sequence \{s[n]\} to determine the beam which captures the highest signal magnitude. The real valued observations are made after synchronization separately for each candidate beam, so that we observe \(r_i[n] = A_i s[n] + w_i[n]\), where \(i \in \{1, \ldots, M\}\), and \(n = 0, \ldots, N - 1\) indicate the beam and sample indices. \(A_i\) is treated as a deterministic unknown amplitude\(^1\) corresponding to beam \(i\) and \(w_i[n]\) is zero mean white Gaussian noise (WGN) with unknown variance \(\sigma^2\). A pseudo-random sequence with \(s[n] \in \{\pm 1\}\), variance one and \(P\{s[n] = +1\} = P\{s[n] = -1\} = 1/2\) is assumed for training so that \(E[s[n]s[n-k]] \approx \delta[k]\) holds for its covariance sequence.

\(^1\)The values \(A_i\) can be restricted to magnitudes as a consequence of the assumption of perfect synchronization. This assumption simplifies the performance analysis of the fixed length test.
Because already the discrimination between two beams and a single plane wave signal (one AoA) shows most essential features of our problem, we restrict the number of candidate beams to \( M = 2 \) and consider a single user scenario that uses correlated observations \( y_i[n] = s[n]r_i[n] \). Considering the difference \( D = A_1 - A_2 \), a sufficient statistic based on the data \( y_i[n] \) equal to the Minimum Variance Unbiased Estimator (MVUE) for \( D \) is given by the sample mean after correlation as

\[
\hat{D} = \frac{1}{N} \sum_{n=0}^{N-1} y_1[n] - y_2[n] = \bar{y}_1 - \bar{y}_2 . \tag{1}
\]

It follows a Gaussian PDF as \( \hat{D} \propto N(A_1 - A_2, 2\sigma^2/N) \). The performance of the test is characterized by the so-called deflection coefficient \( d^2 = N(A_1 - A_2)^2/(2\sigma^2) \).

A conventional fixed length test [3, 5] with arbitrary length \( N \) decides for beam \( i = 1 \), if \( \hat{D} > 0 \) or \( i = 2 \), if \( \hat{D} < 0 \). Defining \( A_{\max} \equiv \max(A_1, A_2) \) this results in the average captured relative magnitude given by

\[
\bar{a} = \frac{A_1}{A_{\max}} \text{P}\{\hat{D} > 0\} + \frac{A_2}{A_{\max}} \text{P}\{\hat{D} < 0\}
\]

\[
= Q\left(\frac{A_2 - A_1}{\sqrt{2\sigma^2/N}}\right) \frac{A_1}{A_{\max}} + Q\left(\frac{A_1 - A_2}{\sqrt{2\sigma^2/N}}\right) \frac{A_2}{A_{\max}}, \tag{2}
\]

where the \( Q \)-function returns the right tail probability of a standard Gaussian r. v. with zero mean and unit variance.

As performance criterion we choose the normalized average loss of signal magnitude denoted by \( \bar{l} \). Defining the ratio \( r \equiv \min(A_1, A_2)/A_{\max} \in [0, 1] \) the value of \( \bar{l} \) achieved with a fixed length test based on the correlation mean can be characterized using Eq. (2) as

\[
\bar{l} = 1 - \bar{a} = (1 - r)Q\left(\frac{d}{\sigma}\right). \tag{3}
\]

The fractional loss \( \bar{l} \) is upper-bounded by \((1 - r)/2\), corresponding to beam selection by coin tossing. Note that for \( r = 1 \) we can not loose anything. Eq. (3) indicates that to achieve a target performance specified by \( \tilde{l}_{\text{target}} \) at given \( r \), the required deflection coefficient is

\[
d_{\text{req}}^2 = \left(\frac{\tilde{l}_{\text{target}}}{1 - r}\right)^2. \tag{4}
\]

From this we see that the minimum required number of observations to achieve a certain value \( \tilde{l}_{\text{target}} \) is

\[
N_{\text{req}} = \frac{2\sigma^2d_{\text{req}}^2}{(A_1 - A_2)^2}. \tag{5}
\]

The problem that arises in the design of the training sequence (i. e. a detector) is to choose a length \( N_{\text{fix}} \) to achieve \( \tilde{l}_{\text{target}} \) in a range of scenarios that occur randomly in applications where exact knowledge about \((A_1 - A_2)^2/\sigma^2\) and \( r \) is not or only roughly available. This will lead to a mismatch between the effectively achieved deflection coefficient and its required value that we describe as

\[
d_{\text{min}}^2 = d_{\text{eff}}^2 - d_{\text{req}}^2 , \tag{6}
\]

where the effective deflection coefficient \( d_{\text{eff}}^2 = N_{\text{fix}}(A_1 - A_2)^2/2\sigma^2 \) will be based on \( N_{\text{fix}} \) used for the design. As illustrated in Fig. 2, the mismatch causes the achieved value of \( \bar{l} \) to deviate from \( \bar{l}_{\text{target}} \). The sensitivity to this mismatch between effective and required deflection coefficients increases as the target value \( \bar{l}_{\text{target}} \) for the fractional loss decreases (a stricter requirements lead to stronger deviations). E. g., for \( \bar{l}_{\text{target}} = 0.01 \), a 3 dB deviation of \( d_{\text{eff}}^2 \) from \( d_{\text{req}}^2 \) leads to values of \( \bar{l} \approx 0.1 \) and \( \bar{l} \approx 5 \times 10^{-4} \) respectively.

Therefore, naively fixing the test length to some value \( N_{\text{fix}} \) based on a certain assumed operating point can result in a strongly variable performance in practical scenarios. Additionally, if \( N_{\text{fix}} \) is conservatively set to a high value based on the worst still acceptable operating point, a lot of time spent will be wasted for detection of the best beam if the channel quality is better than expected.

III. A GENERALIZED LIKELIHOOD RATIO TEST TO DETECT AN UNKNOWN DC LEVEL OR ITS ABSENCE

The building block of our sequential test presented in the next section is the fixed length Generalized Likelihood Ratio Test (GLRT) to detect the presence or absence of a nonzero unknown DC level in zero mean WGN with unknown variance. Consider the composite detection problem

\[
\mathcal{H}_0 : y[n] = w[n] \quad \mathcal{H}_1 : y[n] = A + w[n]
\]

with \( n = 0, \ldots, N - 1 \), \( A \) a nonzero deterministic parameter and \( w[n] \) zero mean WGN with unknown variance \( \sigma^2 \).

Since the variance of the PDF under \( \mathcal{H}_0 \) is not known, a proper threshold to bound the probability of deciding \( \mathcal{H}_1 \) when \( \mathcal{H}_0 \) is true (typically denoted as probability of false alarm \( P_{\text{FA}} \)) can not be found using a simple Neyman-Pearson (NP) approach [3]. Instead one can use the Maximum Likelihood (ML) estimators of the unknown parameters derived from the available observations and insert them into the likelihood functions under each hypothesis [5]. The ML estimates of \( A \) and \( \sigma^2 \) under \( \mathcal{H}_1 \) are \( \hat{A} = \bar{y} \) (the sample mean) and

\[
\hat{\sigma}^2_{\mathcal{H}_1} = (1/N) \sum_n (y[n] - \bar{y})^2
\]

while under \( \mathcal{H}_0 \) the ML estimate

\[
\hat{\sigma}^2_{\mathcal{H}_0} = (1/N) \sum_n (y[n] - \bar{y})^2
\]
of $\sigma^2$ is just $\hat{\sigma}_N^2 = (1/N) \sum_i y[i]^2$. By replacing the unknown parameters with their estimates in the Gaussian PDFs under both hypotheses the GLR can be calculated as

$$L_G(y) = \frac{p(y: \hat{A}, \hat{\sigma}_N^2, H_1)}{p(y: \hat{\sigma}_N^2, H_0)} = \left( \frac{\hat{\sigma}_N^2}{\sigma_H^2} \right)^{\frac{N}{2}}. \tag{6}$$

Eq. (6) indicates that deciding for $H_1$ makes sense when the fit of data to a signal amplitude $A = y$ produces a much smaller error, as measured by $\hat{\sigma}_N^2$, compared to a fit to the no signal hypothesis reflected by the estimate $\hat{\sigma}_N^2 = \hat{\sigma}_H^2 + \hat{y}^2$.

The remaining task is to find a proper decision threshold in order to bound $P_{FA}$. To this end let us define a random variable $\gamma$ as $\gamma = 2 \ln L_G(y)$. A non-trivial result that can be found in [5] states that for large $N$ the variable $\gamma$ follows either a central or a non-central $\chi^2$-distribution with one degree of freedom. Therefore asymptotically in $N$ it holds that

$$\gamma = N \ln \left( 1 + \frac{\hat{y}^2}{\hat{\sigma}_H^2} \right) \sim \begin{cases} \chi_1^2, & \text{under } H_0 \\ \chi_1^2(\lambda), & \text{under } H_1 \end{cases}. \tag{7}$$

The ratio $\lambda = N(A^2/\sigma^2)$ that plays the role of the deflection coefficient is denoted in statistics as the non-centrality parameter of the $\chi^2$-PDF. Using the NP design criterion, we can now ensure that $P_{FA}$ will not surpass a predefined value by finding a proper threshold $\gamma_{th}$. Noting that a $\chi_1^2$ r.v. $\gamma$ is related to a standard normal r.v. $x \sim N(0,1)$ as $\gamma = x^2$, it follows that $P_{FA} = \Pr\{\gamma > \gamma_{th}; H_0\}$ can be expressed as a sum of $Q$-functions as $P_{FA} = \Pr\{x > \sqrt{\gamma_{th}}\} + \Pr\{x < -\sqrt{\gamma_{th}}\} = 2Q(\sqrt{\gamma_{th}})$. This leads to

$$\gamma_{th} = \left[ Q^{-1}\left( \frac{P_{FA}}{2} \right) \right]^2. \tag{8}$$

Similarly, the second type of error, the probability of misdetection $P_{MD} = 1 - \Pr\{\gamma > \gamma_{th}; H_1\}$ becomes

$$P_{MD} = Q(\sqrt{\lambda} + \sqrt{\gamma_{th}}) - Q(\sqrt{\lambda} - \sqrt{\gamma_{th}}). \tag{9}$$

As shown in Fig. 3, the probability of misdetection $P_{MD}$ decreases exponentially as $N$ increases due to the $Q$-function. In addition, the rate of decrease depends strongly on $A^2/\sigma^2$. Therefore large performance fluctuations will occur for fixed $N$ that can only be avoided by a variable length test. The sequential competition test that we introduce next exploits the strong dependence of $P_{MD}$ on $A^2/\sigma^2$ and $N$, and turns it into an advantage.

IV. Variable Length Test with Unknown Parameters

Consider again the initial $M$-ary decision problem stated in Section II where separate observation sequences are available under each beam and the aim is to detect the beam that captures the highest signal magnitude. But instead of comparing the estimates of the values $\{A_1, \ldots, A_M\}$ against each other, let us rather compare them separately to the absence of a signal. This means that under each beam we formulate the same binary hypothesis test stated in the previous section as

$$H_0 : y[i] = w_i[n] \quad \text{vs.} \quad H_1 : y[i] = A_i + w_i[n],$$

where as before $A_i$ for $i = 1, 2, \ldots, M$ are nonzero deterministic unknowns and the $w_i[n]$ are zero mean WGN samples with equal unknown variance $\sigma^2$. However, $n = 0, 1, \ldots,$ now can grow until a decision criterion is fulfilled. Obviously, $H_0$ is the wrong hypothesis under each beam, assuming that some signal is observable but with different strength. On the other hand it acts as a common reference in the set of all composite hypothesis tests.

Let us denote the probability of correct decision under beam $i$ after $n$ observations as $P_{D_{ij}}(n)$. Then it follows from Eq. (8) that for $|A_i| > |A_j|$ and a common decision threshold $\gamma_{th}$ that $P_{D_{ij}}(n) > P_{D_{ji}}(n)$. This is simply a consequence of the trivial fact that the accumulated deflection coefficient $\gamma_{ij} = nA_i^2/\sigma^2$ will grow more quickly than $\gamma_{ji} = nA_j^2/\sigma^2$. As a consequence, if we use the random variable $\gamma_i(n)$ introduced in the last section as a decision metric, the beam that observes the stronger signal will cross the threshold earlier on average.

For ease of exposition of this phenomenon and an initial numerical performance evaluation let us return to the case of two beams which leads to the following sequential test applied on $\gamma_i(n)$ for $i = 1, 2$:

$$\begin{cases} \text{beam}_1 : \gamma_1(n) = n \ln(1 + \frac{\hat{y}_1^2}{\sigma_{\gamma_{1i}}^2}) & H_i \geq \gamma_{th} \\ \text{undecided} & H_i < \gamma_{th} \end{cases} \quad \begin{cases} \text{beam}_2 : \gamma_2(n) = n \ln(1 + \frac{\hat{y}_2^2}{\sigma_{\gamma_{2i}}^2}) & H_i \geq \gamma_{th} \\ \text{undecided} & H_i < \gamma_{th} \end{cases}. \tag{10}$$

The sequences $\gamma_i(n)$ are stochastic trajectories over $n$ corresponding to beams $i = 1, \ldots, M$, that are compared to the fixed common threshold $\gamma_{th}$ at each observation $n$. The test terminates as soon as one of the trajectories surpasses the threshold while the index of this trajectory indicates the selected beam. Otherwise we continue by taking the next observation into account. The interpretation is that we let the beams compete to distinguish themselves from pure zero mean WGN with unknown variance, and the one which does it faster is the winning beam in the competition, see Fig. (4).

Characterizing the performance again in terms of $\bar{l}$ we observe in Fig. 5 (top) that the sequential competition test shows
Fig. 4. Sequential competition test visualized for \((A_1 - A_2)^2 / \sigma^2 = -10\) dB and \(r = 0.1\). The threshold \(\gamma_{th}\) based on \(P_{FA} = 10^{-3}\) is indicated by dashed lines and first passages over \(\gamma_{th}\) are depicted via circles. Different colors represent 5 different realizations of the competition test.

Fig. 5. Achieved \(\bar{l}\) using a fixed length test in comparison to the achieved \(\bar{l}\) using the sequential test with equal \(\gamma_{th}\) based on \(P_{FA} = 10^{-3}\).

an essentially invariant performance of \(\bar{l}\) for varying value of the true deflection coefficient. This is in strong contrast to the fixed length test that was designed for some SNR operating point (\(l_{\text{target}} = 0.03\) at \((A_1 - A_2)^2 / \sigma^2 = -7\) dB, \(r = 0.5\)). The reason is that the required average test length \(\bar{n}\) of the sequential competition test is changing adaptively as a function of \((A_1 - A_2)^2 / \sigma^2\) as shown in the bottom plot. The other benefit of the sequential competition test is that it requires on average less observations \(\bar{n}\) in the lower SNR regime to achieve a certain value of \(l_{\text{target}}\) than the required number \(N_{\text{req}}\) of a fixed length test with genie knowledge that would achieve the same \(l_{\text{target}}\) (see Fig. 5, bottom). This can be understood intuitively, because according to the fluctuations of the competing stochastic processes around their typical behaviour, the test exactly terminates when a reliable discrimination becomes possible, so that the test terminates earlier on average (which was the original motivation of Wald to develop his test [4]). This property is of particular interest, because the competition test reduces training time at exactly those smaller values of \((A_1 - A_2)^2 / \sigma^2\) where many observations are needed. On the other hand, when \((A_1 - A_2)^2 / \sigma^2\) becomes large the detection problem becomes easy and we do not need many observations in the first place.

V. Beam Selection in Massive MIMO Systems

We numerically studied the sequential competition test in a small scale massive MIMO scenario [8] using the codebook of a Butler matrix [6] with 16 beams of a uniform linear array considering a single path channel (see Fig. 1 where magnitude patterns for 8 beams are shown). The AoA was distributed uniformly in \([-90^\circ, 90^\circ]\) over the simulation runs while SNR was defined as \(A_{\text{max}}^2 / \sigma^2\) indicating the maximum available SNR of the best beam. The quantities \(l\) and \(\bar{n}\) were estimated at each SNR point based on \(10^4\) simulation runs for SNR values in the interval \([-6, 6]\) dB. For comparison we designed a fixed length test on the design criterion that \(\bar{l}\) should not surpass \(\bar{l} = 0.15\) (\(\bar{l}\) to power loss of 1.4 dB) even at the lowest SNR of interest (here \(A_{\text{max}}^2 / \sigma^2 \geq -6\) dB which lead to \(N_{\text{fix}} = 50\)).

As shown in Fig. 6, the sequential competition test with the same \(\gamma_{th}\) for all beams based on \(P_{FA} = 10^{-3}\) keeps \(\bar{l}\) in an interval of a few percent around some mean value of \(\lesssim 10\%\) considered to be sufficient in practice, while adaptively decreasing the average test length \(\bar{n}\) as \(A_{\text{max}}^2 / \sigma^2\) grows larger. This results in reduced delay due to training at SNR values above \(-6\) dB. The achievable speed up becomes even more interesting in systems using analogue or hybrid beamforming [7], where exhaustive search for beam selection can cause huge delays as the codebook size increases further and/or codebooks at both ends of the link need to be trained.

VI. Concluding Remarks

We propose a novel sequential hypothesis test based on GLR statistics to solve the composite beam selection problem. Our sequential competition test shows properties such as adaptivity w.r.t. the SNR operating point and speeds up beam selection even when compared to an optimally designed fixed length test at lower SNR where it matters the most or most time may be lost for training in a possible application. These properties can be of interest in massive MIMO systems using hybrid beamforming as well as under conditions where the training time is limited due to small channel coherence time.
REFERENCES


