Fractionally-Spaced Prefiltering for Reduced State Equalization

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Abstract — Reducing the number of states is a possible solution for complexity reduction of trellis-based equalization. It has been often proposed that such approaches profit from a near minimum phase target channel which may be obtained by prefiltering the incoming sequence appropriately. A possible finite length solution of such a prefilter is given by the optimal settings of the decision feedback equalizer with a finite length constraint on its feedforward and feedback filter. In this paper, we focus on the limiting behavior of this solution with respect to the unconstrained solution which is obtained by spectral factorization.

1 Introduction

In the presence of intersymbol interference (ISI), trellis-based equalizers (TBE) with help of the Viterbi algorithm, the BCJR algorithm [1], and others compute optimal estimates with respect to their criteria. The drawback of those TBE is their large complexity, which grows exponentially with the channel memory \( q \) of the underlying ISI channel. For complexity reduction, a powerful concept (among others) is to combine trellis states into hyper states and to apply feedback techniques on the trellis [2]. Such reduced state approaches profit from a near minimum phase target channel. Thus, the objective of prefiltering the incoming sequence is to modify the energy profile of the resulting target channel rather than obtaining a truncated channel of memory \( q' < q \), see [3].

2 Base-Band Model

We assume the usual simplified equivalent base-band model for digital pulse amplitude modulation (PAM). Let

\[
s(t) = \sum_{k=-\infty}^{\infty} x(k) p_s(t - kT), \quad t \in \mathbb{R},
\]

be the continuous base-band PAM-signal with symbol period \( T \). The random data sequence \( x = \{x(k)\}_{k \in \mathbb{Z}} \) is i.i.d., white and zero mean with variance \( \sigma_x^2 = \mathbb{E}\{|x(k)|^2\} = 1 \), where \( x(k) \) is element of some appropriately scaled \( Q \)-ary complex set \( \mathcal{A} = \{a_0, a_1, \ldots, a_{Q-1}\} \subset \mathbb{C} \). The transmit filter \( p_s \) has norm \( \|p_s\|_2 = \sqrt{T} \). The signal \( s \) passes through a linear channel \( c = c(t) \) which is assumed to be time-invariant for simplicity (low vehicle speed). It is a realization of the WSSUS process

\[
C(t) = \frac{1}{\sqrt{N_c}} \sum_{i=0}^{N_c-1} e^{j\theta_i} \delta(t - \tau_i)
\]

generated by i.i.d. random variables \( \theta_i \) and \( \tau_i \) for \( i = 0, \ldots, N_c - 1 \), where \( N_c \) is a large number (\( N_c \approx 100 \)); refer to [5] and the references therein for some background information on this statistical model. \( \theta_i \) is uniformly distributed over \( [-\pi, \pi) \) and \( \tau_i \) is distributed according to \( p_{\tau_i}(\tau) = \sum_{k=1}^{K} p^{(k)}(\tau) \) with

\[
p^{(k)}(\tau) = \begin{cases} 2^{-k} e^{-(\tau - \tau_{s,k})} \frac{a_k}{\tau_{s,k}} & \tau \leq \tau_{s,k}, \\ 0 & \text{otherwise,} \end{cases}
\]

where \( K, a_k, b_k, \tau_{s,k} \) and \( \tau_{s,u} \) are parameters. Let \( p_r \) be a \( \sqrt{N_0} \) Nyquist receive filter at the receiver.

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With \(-2\) we define the Hermitian covariance matrix
\begin{equation}
R_{\lambda,M,t} = \begin{pmatrix} h_0 & h_1 & \cdots & h_q & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_q & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & h_0 & h_1 & \cdots & h_q \\
\end{pmatrix}
\end{equation}
(6)

With \[ \lambda = \frac{\sigma^2}{\sigma^2_n}, \]
we define the Hermitian covariance matrix
\begin{equation}
R_{\lambda,M,t} = (\lambda I_{M+q} + H_{M,t}H_{M,t}^*) \in \mathbb{C}^{(M+q)\times(M+q)}.
\end{equation}

Let\(^3\) \(R_{\lambda,M,t} = LDL^*\) be the Cholesky factorization of \(R_{\lambda,M,t}\), where \(L\) is monic, lower triangular and \(D = \text{diag}(d_1, \ldots, d_{M+q})\) is diagonal. Let \(\mathbf{e}_i \in \mathbb{R}^{M+q}\) be the \(i\)-th unit vector, which has a one at the \(i\)-th position and zeros in all other positions. Then
\begin{equation}
b_{\lambda,M,t} = L\mathbf{e}_M = (0, \ldots, 0, 1, b_{\lambda,M,t}(1), \ldots, b_{\lambda,M,t}(q))^T,
\end{equation}
and
\begin{equation}
w_{\lambda,M,t} = H_{M,t}D^{-1}L^{-*}\mathbf{e}_M = (w_{\lambda,M,t}(-\ell M + 1), \ldots, w_{\lambda,M,t}(0))^T
\end{equation}
are the optimal and unique FIR-MMSE-DFE filter settings under certain conditions [7], depending on \(\lambda > 0\), \(M > 0\) and \(\ell\). The (strictly) causal \(T\)-spaced feedback filter has the form \(\{\delta(k) - b_{\lambda,M,t}(k)\}_{k \in \mathbb{Z}}\), where \(b_{\lambda,M,t} = \{1, b_{\lambda,M,t}(1), \ldots, b_{\lambda,M,t}(q)\}\) and the anti-causal \(T/\ell\)-spaced feedforward filter has the response \(w_{\lambda,M,t} = \{w_{\lambda,M,t}(-\ell M + 1), \ldots, w_{\lambda,M,t}(0)\}\). For the computation of \(b_{\lambda,M,t}\) and \(w_{\lambda,M,t}\), efficient generalized Schur-type algorithms of complexity \(O(\ell(M + q)^2)\) exist [8], which have been recently improved by the authors [9]. Fractionally spaced filtering of \(z^{(3)}\) with \(w_{\lambda,M,t}\) and additional downsampling by \(\ell\) yields a target system
\[ r(k) = (x \ast b_{\lambda,M,t})(k) + n(k), \quad k \in \mathbb{Z}, \]
where the noise sequence \(n\) is neither white nor Gaussian in general. However, for \(\lambda = \sigma^2_n/\sigma^2\), the noise variance is exactly given by
\[ \sigma^2 = \text{E}\{[n(k)]^2\} = \frac{\sigma^2_n}{M}, \]
see [7]. The monic causal target channel \(b_{\lambda,M,t}\) is not necessarily minimum phase. It can be shown, that for \(\lambda > 0\)
\[ \lim_{M \to \infty} b_{\lambda,M,t} = g_{\lambda,t} \quad \text{and} \quad \lim_{M \to \infty} w_{\lambda,M,t} = \psi_{\lambda,t}, \]
using results of the unconstrained MMSE-DFE [10]. Hence, the limit \(\lambda \to 0\) of the optimal prefilter of the unconstrained MMSE-DFE appears to approach the desired whitened matched filter. As it has already been proposed in [3], using the FIR MMSE-DFE solution with respect to near minimum phase prefiltering, one might, therefore, consider \(\lambda\) as a free parameter, rather than choosing \(\lambda\) according to (7). However, the limiting behavior \(b_{\lambda,M,t} \to g_{\lambda,t}\) can be strongly influenced by \(\lambda\), as shown in the next section.

### 5 Limiting Behavior

The element of the \(i\)-th row and \(j\)-th column of \(R_{\lambda,M,t}\) for \(i, j = 1, \ldots, M + q\) is given by:
\begin{equation}
R_{\lambda,M,t}[i, j] = \lambda \delta(i-j) + \sum_{m=0}^{\ell-1} \sum_{r=0}^{M} h_t^{(m)}(i-t)(h_t^{(m)}(j-t))^*,
\end{equation}
(11)
It can be easily seen by inspection of equation (11) that
\begin{equation}
R_{\lambda,M,t} - Z_{M+q}R_{\lambda,M,t}Z_{M+q}^* = G_{\lambda,M,t}J_{t}G_{\lambda,M,t}^* + \sum_{m=0}^{M} h_t^{(m)}(i-t)(h_t^{(m)}(j-t))^*,
\end{equation}
(12)
where \(Z_{M+q}\) is the lower shift matrix of dimension \((M + q) \times (M + q)\), i.e., it has ones on its first lower sub-diagonal and zeros everywhere else. This so called displacement yields a generator \((J_{t}, G_{\lambda,M,t})\) of \(R_{\lambda,M,t}\) where \(J_{t} = (I_{t+1} \otimes I_{t})\) and \(G_{\lambda,M,t} = \{g_t^1, \ldots, g_t^{\ell+1}, g_t^0, \ldots, g_t^0\} \in \mathbb{C}^{(M+q) \times (2\ell+1)}\), with
\[ q_t^i = \begin{cases} \sqrt{\lambda} \mathbf{e}_1 = \sqrt{\lambda}(1, 0, \ldots, 0)^T \quad & i = 1 \\ (h_t^{(i-2)}(0), \ldots, h_t^{(i-2)}(q), 0, \ldots, 0)^T \quad & i = 2, \ldots, \ell + 1 \\ (0, \ldots, 0, h_t^{(i-1)}(0), \ldots, h_t^{(i-1)}(q-1))^T \quad & i = 1, \ldots, \ell. \\ \end{cases} \]
(13)
We can obtain a semi-infinite extension \(R_{\lambda,\infty,t}\) of \(R_{\lambda,M,t}\) by letting \(M \to \infty\). The covariance generating function [11] associated with \(R_{\lambda,\infty,t}\) is defined as
\[ \mathcal{P}_{\lambda,t}(z, w) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} R_{\lambda,\infty,t}[i+1, j+1]z^i(w^*)^j, \]
frequency is $\pi/T$ and $\pi\ell/T$, respectively. Truncation of $h_\ell$ is obtained by choosing $q_1 = 1$ and $q_0 = 4$. Ideal frequency hopping between consecutive data bursts is modeled by computing a new physical channel realization $c$ (i.e., a new set of variables $\theta_i$ and $\tau_i$ for $i = 0, \ldots, N_c - 1$) for each burst [5]. With $h_\ell$, the prefilter is computed anew, given $M$ and $\lambda$. Now, let $M = 8$ and $\lambda = 0.04$. Figure 3 plots the bit error rate (BER) versus the ratio $E_b/N_0$, where $E_b = E_s/R_c$. It can be seen that the $T/2$-spaced system performs slightly better than the $T$-spaced system. Choosing $\lambda = 0.001$, both systems degrade, but the $T$-spaced system performs significantly worse. However, the comparison might not be fair, since the prefilter for the $T$-spaced case requires only $M$ taps. Extending $M$ to 16 revealed that the $T$-spaced prefilter performs less sensitive regarding the choice of $\lambda$. Further simulations indicated that $\lambda = \sigma_n^2/\sigma_s^2$ is almost optimal. The robustness with respect to $\lambda$ is important for practical reasons, since estimation of $\sigma_s^2$ is not trivial.

6 Conclusions

Computing the optimal FIR MMSE-DFE settings offers an attractive solution for a finite length prefilter which targets an overall near minimum phase channel. It has been demonstrated that a fractionally spaced prefilter is theoretically more robust than a symbol spaced prefilter with respect to parameter mismatch. The difference appears to be less severe for the BU channel model, if the prefilter length is sufficiently long.

References


