Optimal Antenna Positioning for Wireless Board-To-Board Communication Using a Butler Matrix Beamforming Network

Johannes Israel, John Martinovic, and Andreas Fischer  
Institute of Numerical Mathematics  
Technische Universität Dresden  
01062 Dresden, Germany  
Email: {Johannes.Israel, Andreas.Fischer}@tu-dresden.de, John.Martinovic@mailbox.tu-dresden.de

Michael Jenning and Lukas Landau  
Communications Laboratory  
Technische Universität Dresden  
01062 Dresden, Germany  
Email: Michael.Jenning@tu-dresden.de, Lukas.Landau@ifn.et.tu-dresden.de

Abstract—For wireless board-to-board communication the problem of an optimal positioning of antenna arrays on two boards facing each other is described, where each antenna array uses the same kind of Butler matrix beamforming network. We discuss possible design criteria for the positioning problem. To obtain a position with both a high response and a low pathloss for as many as possible links we introduce a corresponding optimization problem and show how it can be solved.

I. INTRODUCTION

In a wireless board-to-board communication scenario, let two parallel boards with distance $d$ be equipped with the same number $N$ of antenna arrays (nodes). Thus, the theoretical number of possible wireless links between the nodes on the different boards is $N^2$. The uniform $M \times M$ antenna arrays are assumed to be switched beam antennas with fixed beams determined by a two-stage Butler matrix beamforming network. We assume that each node uses the same kind of beamforming network. This network can generate $M^2$ different spatial beams. Our aim is to find an optimal positioning of nodes w.r.t. a reasonable and tractable optimization criterion. Based on the calculation of the response of all possible beams that can be generated by the beamforming network, we first provide a condition under which a beam has the highest possible response. Under all links with highest possible response we suggest to maximize the number of those links with shortest distance. For any number $N$ of nodes on each of the two boards and for the common case that $M = 4$, an optimal positioning of these nodes will be derived.

II. BACKGROUND

An application for the beamforming based wireless board-to-board communication (see Fig. 1) is given by the replacement of copper-based connections in a super-computer environment with massive parallelization, see the project “Ultra High-Speed Wireless Board-to-Board Communication” within the Collaborative Research Center “Highly Adaptive Energy-efficient Computing” (HAEC) [1]. Besides the reduction of required material, a reduced routing complexity and an increased flexibility are some of the main benefits. Providing equivalent connections as compared to the copper-based transmission with data rates of several multigigabit/second requires a huge bandwidth and therefore also extremely high carrier frequencies up to the lower terahertz region. Taking into account the energy consumption for high-speed analog-to-digital converters leads to the conclusion that digital beamforming cannot be applied in such a system. Furthermore, the design of voltage controlled phase shifters for analog beamforming might be challenging, such that this work considers the utilization of Butler matrix beamforming networks and especially the optimal positions of the wireless devices placed on the boards. This approach is promising as the board-to-board scenario can be assumed as static.

In general, there are numerous possibilities to achieve adaptively steered beams. The simpler forms are switching networks, such as the Butler matrix [2], Rotman lens [3] or Blass matrix [4]. Their advantage is that they can be realized using passive circuit elements only. On the other hand, they only offer a limited number of fixed beams. Increasing the number of beams also drastically increases the network complexity. The increase in beam number is often only achieved through the increase of antenna elements. Furthermore, beam shaping capabilities are limited.

Utilizing analog or discrete phase shifters is another possibility to achieve beam switching or even beam steering. For discrete phase shifters the appropriate discrete beamforming problem has been investigated in [5]. Analog beam shifters...
with full 360° range offer the greatest freedom as they ideally can be adjusted independently from the amplitudes. This allows full beam shaping and beam steering capabilities. The cost of this feature is the requirement of active circuit elements. Especially in the higher mm-wave frequency range above 60 GHz, those elements are not readily available, especially when limited to a certain semiconductor technology. Their development is a laborious task and subject of another project within HAEC.

Because of the limited number of fixed beams in switching networks, phase shifter based beam steering networks are in general superior when full adaptivity is required. Since HAEC addresses server computers, it is possible to fix the layout of computer boards w.r.t. the wireless communication nodes, like processors or memory. As it will be shown in this paper, it is possible to find an optimal placement of such nodes w.r.t. the processors or memory. As it will be shown in this paper, it is possible to find an optimal placement of such nodes w.r.t. the fixed beams. This ensures the successful usage of fixed beam switching networks instead of phase shifters. Additionally, a simplification and relaxation of the active circuitry is achieved, since the beam switching network can be realized with passive components only.

The Butler matrix was chosen as it is compatible both to conventional printed circuit board technology as well as to on chip integration. A Rotman lens would require a large and solid copper area that is not compatible to on chip design rules. Compared to a Butler matrix, a Blass matrix requires more circuit elements. First designs of a Butler matrix used within HAEC have already been presented [6].

III. BUTLER MATRIX BEAMFORMING NETWORKS

In this section we provide some basic facts on Butler matrix beamforming networks for linear and quadratic antenna arrays.

A. Butler matrix for a linear antenna array

Firstly, we consider a uniform linear antenna array with a distance of \(d_x = \lambda/2\) between the antenna elements, where \(\lambda\) is the signal’s wavelength. In Fig. 2 a schematic of a 4 × 4 Butler matrix for a linear antenna array is depicted. The Butler matrix consists of two fixed 45° phase shifters and four passive four-port hybrid power dividers. Each output \(O_1, \ldots, O_4\) depends on the inputs \(I_1, \ldots, I_4\). Independent from the mode of operation (receive or transmit), we define the outputs to be the interface between the Butler matrix and the antenna. The inputs are the interface between the Butler matrix and the front end. In operation, just one input, or frontend, and all four outputs, or antennas, are active. The inactive inputs have to be properly terminated. The outputs can be described as

\[
\begin{pmatrix}
O_1 \\
O_2 \\
O_3 \\
O_4
\end{pmatrix} = \begin{pmatrix}
e^{j\frac{\pi}{4}} & e^{j\frac{3\pi}{4}} & e^{j\frac{\pi}{4}} & e^{j\frac{\pi}{4}} \\
e^{j\frac{\pi}{4}} & e^{j\frac{\pi}{4}} & e^{j\frac{3\pi}{4}} & e^{j\frac{3\pi}{4}} \\
e^{j\frac{3\pi}{4}} & e^{j\frac{\pi}{4}} & e^{j\frac{\pi}{4}} & e^{j\frac{\pi}{4}} \\
e^{j\frac{\pi}{4}} & e^{j\frac{3\pi}{4}} & e^{j\frac{3\pi}{4}} & e^{j\frac{\pi}{4}} \\
\end{pmatrix} \begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{pmatrix}
\]

In the general case for a linear antenna array, the elements of an \(M \times M\) Butler matrix \(B = (b_{mk}) = [b_1, \ldots, b_M]^\top\), \(m, k \in \{1, \ldots, M\}\), can be described as (see also [7])

\[
b_{mk} = \frac{1}{M} e^{j\pi k(2m-1)/M},
\]

where the parameter \(M\) is usually a power of 2. The vector \(b_m\) corresponds to the choice of the \(m\)-th input port of the Butler matrix beamforming network. We briefly show how to derive the main response axis (MRA) of the antenna array depending on the input port of the given Butler matrix. Without loss of generality, we only consider the transmit case here. To get the output signal of the antenna array, the Butler matrix and the antenna array configuration have to be superimposed. For a signal \(s\) the output signal for the \(m\)-th input port is

\[
y = b_m^\top a s,
\]

where \(a \in \mathbb{C}^M\) is the array steering vector. The steering vector depends on the spatial angle \(\theta\) (measured to the array perpendicular) and is given by

\[
a(\theta) = \left(1, \ e^{-2j\pi d_x \sin(\theta)/\lambda}, \ldots, \ e^{-2(M-1)j\pi d_x \sin(\theta)/\lambda}\right)^\top.
\]

The array response \(R_m(\theta)\) for the \(m\)-th port and a direction \(\theta\) then becomes

\[
R_m(\theta) = \left|b_m^\top a(\theta)\right|^2 = \begin{cases} 
1 & \text{for } \phi_m = 2n\pi, n \in \mathbb{Z}, \\
\sin^2(\phi_m)/\sin^2(\phi_m) & \text{else},
\end{cases}
\]

with

\[
\phi_m = \frac{\pi}{2} \left(\frac{1}{M} (2m-1) - \sin(\theta)\right).
\]

We can derive the main response axis (MRA) for the beamformer \(b_m\) from the equation \(\phi_m = 2n\pi\) and obtain

\[
\theta^*_m = \begin{cases} 
\arcsin\left(\frac{2m-1}{M}\right) & \text{for } m = 1, \ldots, \frac{M}{2}, \\
\arcsin\left(\frac{2m-1}{M} - 2\right) & \text{for } m = \frac{M}{2} + 1, \ldots, M.
\end{cases}
\]

Combining all beampatterns for every beamformer \(b_m\) with \(m \in \{1, \ldots, M\}\) in one plot we get the result depicted in Fig. 3 for \(M = 4\) and \(M = 8\).
B. Butler matrix for a quadratic antenna array

Now, the introduced concept is transferred from uniform linear antenna arrays to uniform quadratic antenna arrays with the distance \(d_x = \lambda/2\) between the antenna elements in \(x\)- and \(y\)-direction. Creating spatial beams with a Butler matrix beamforming network has been dealt with in [8] and [9], for example. In what follows, we assume that the beamforming network for the uniform \(M \times M\) antenna array is described by the Kronecker product of two Butler matrices. Let \(B\) be the \(M \times M\) Butler matrix from (1). Then the Butler matrix \(F\) for an \(M \times M\) uniform antenna array is

\[
F := B \otimes B = \begin{pmatrix}
    b_{11} B & \cdots & b_{1M} B \\
    \vdots & \ddots & \vdots \\
    b_{M1} B & \cdots & b_{MM} B
\end{pmatrix}.
\]

This two-stage Butler matrix beamforming network generates \(M^2\) different spatial beams. For \(m, n \in \{1, \ldots, M\}\) the beamformer \(b_{mn}^*\) is given by the \((4(m-1) + n)\)-th column of \(F^T\).

C. MRA for spatial Butler beamforming network

Extending the model given in Section III-A to the quadratic antenna array, the array response becomes a function of two angles, the azimuthal angle \(\phi\) and the elevation angle \(\theta\):

\[
R_{m,n} = \begin{cases}
    1 & \text{for } \alpha_m, \beta_n \in \Pi, \\
    \frac{1}{M^2} \left( \frac{\sin(M\beta_n)}{\sin(\beta_n)} \right)^2 & \text{for } \alpha_m \in \Pi, \beta_n \notin \Pi, \\
    \frac{1}{M^2} \left( \frac{\sin(M\alpha_m)}{\sin(\alpha_m)} \right)^2 & \text{for } \alpha_m \notin \Pi, \beta_n \in \Pi, \\
    \frac{1}{M^2} \left( \frac{\sin(M\alpha_m)}{\sin(\alpha_m)} \right) \left( \frac{\sin(M\beta_n)}{\sin(\beta_n)} \right)^2 & \text{else},
\end{cases}
\]

where \(\Pi := \{k\pi | k \in \mathbb{Z}\}\) and

\[
\alpha_m := \frac{\pi}{2} \left( \frac{1}{M} (2m-1) - \sin(\theta) \cos(\phi) \right),
\]

\[
\beta_n := \frac{\pi}{2} \left( \frac{1}{M} (2n-1) - \sin(\theta) \sin(\phi) \right).
\]

Thus, the maximum response \(R_{m,n}(\theta,\phi) = 1\) is reached if and only if \(n_1, n_2 \in \{0, 1\}\) exist, so that

\[
\alpha_m = \pi n_1 \text{ and } \beta_n = \pi n_2.
\]

**Proposition 1.** For \(n_1, n_2 \in \{0, 1\}\) and \(m, n \in \{1, \ldots, M\}\), there are \(\theta, \phi \in \mathbb{R}\) so that (2) is satisfied if and only if

\[
f_m^2 + g_n^2 \leq 1
\]

with

\[
f_m := \frac{2m - 1}{M} - 2n_1, \quad g_n := \frac{2n - 1}{M} - 2n_2.
\]

**Proof:** Obviously, System (2) is equivalent to

\[
\sin \theta \cos \varphi = f_m, \quad \sin \theta \sin \varphi = g_n.
\]

1) Let \((\theta^*, \varphi^*)\) be a solution of System (4) and (5). Then it holds

\[
f_m^2 + g_n^2 = \sin^2(\theta^*) \cos^2(\varphi^*) + \sin^2(\theta^*) \sin^2(\varphi^*) = \sin^2(\theta^*) (\cos^2(\varphi^*) + \sin^2(\varphi^*)) = \sin^2(\theta^*) \leq 1.
\]

2) Let (3) be satisfied. Then, noting that \(f_m \neq 0\),

\[
\theta^* := \text{sgn}(f_m) \cdot \arcsin \left( \frac{f_m}{f_m} + \frac{g_n}{f_m} \right), \quad \varphi^* := \arctan \left( \frac{g_n}{f_m} \right).
\]

are well-defined with \(\theta^*, \varphi^* \in (-\frac{\pi}{2}, \frac{\pi}{2})\). We emphasize that

\[
\sin(\theta^*) = \frac{\text{sgn}(f_m) \cdot \sqrt{f_m^2 + g_n^2}}{f_m},
\]

Since

\[
\cos \alpha = \frac{1}{\sqrt{1 + \tan^2(\alpha)}} \quad \text{for } \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})
\]

and

\[
\arctan (\alpha) = \arcsin \left( \frac{\alpha}{\sqrt{1 + \alpha^2}} \right) \quad \text{for } \alpha \in \mathbb{R}
\]

we obtain by means of (8), (9) and (10)

\[
\sin(\theta^*) \cos(\varphi^*) = f_m \sqrt{1 + \left( \frac{g_n}{f_m} \right)^2} \frac{1}{\sqrt{1 + \left( \frac{g_n}{f_m} \right)^2}} = f_m.
\]
and
\[ \sin(\theta^\star) \sin(\varphi^\star) = f_m \sqrt{1 + \left(\frac{g_n}{f_m}\right)^2 \frac{g_n}{f_m}} \]
\[ = g_n. \]
Hence, System (4) and (5) has a solution.

**Remark 1.** Condition (3) in Proposition 1 is not necessarily satisfied for each beamformer \( b_n^b \). As an example for \( M = 4 \) a response of 1 can not be reached for \( b_2^b, b_3^b, \) and \( b_4^b \) so that only 12 of the 16 mainlobes achieve the maximal response.

### IV. OPTIMAL POSITIONING OF NODES

We now consider the announced problem of positioning \( N \) nodes on each of two parallel boards with distance \( d \). In the whole section the lower and upper board are located in the planes \( z = 0 \) and \( z = d \), respectively. All the nodes are assumed to be uniform quadratic antenna arrays which use the introduced Butler beamforming network for transmitting and receiving, see Section III-B. Moreover, we want to consider a system where the far field assumption is valid. Now we discuss two criteria for an optimal positioning of nodes. For one of the criteria we will then show how an exact solution can be obtained.

#### A. Total Link loss minimization

If we focus on a single link between two nodes mainly two parameters are of interest, the pathloss and the array response. For two arbitrary positions \( y, z \in \mathbb{R}^3 \) of nodes on two neighboring boards we define the total link loss as
\[ T(y, z) = L(y, z) - 2 \max_{m,n} \left\{ R_{m,n}^B(\theta(y, z), \varphi(y, z)) \right\}, \quad (11) \]
where
\[ L(y, z) = 20 \log_{10} \left( \frac{4 \pi |y - z|}{\lambda} \right) \]
approximates the pathloss and
\[ R_{m,n}^B = 10 \log_{10} R_{m,n} \]
describes the array response for the Butler beamformer \( b_n^b \) in dB. Obviously, the pathloss will be minimized if the two nodes are placed opposite to each other. But since the mainlobe cannot be steered to an elevation angle of \( 0^\circ \) using the proposed Butler beamforming network this positioning would not be optimal with respect to the total loss. Instead of that we would expect that a positioning, where the nodes lie on (or close to) an MRA of each other, is reasonable and at least nearly optimal with respect to total loss. We emphasize that the minimization of the total loss function (11) for a single link is already a nonconvex problem. A criterion for optimal positioning of \( N \) nodes on each board is the minimization of the (weighted) sum of the total link losses for all possible links. This approach would lead to the following optimization problem:
\[
\min_{y',z' \in \mathbb{R}^3} \sum_{i=1}^{N} \sum_{j=1}^{N} T(y', z')
\]
\[ \text{s.t. } y'_i = 0, \quad z'_j = d \quad \text{for all } i, j \in \{1, \ldots, N\}, \]
\[ |y' - y|^2 \geq d_n, \]
\[ |z' - z|^2 \geq d_n \quad \text{for all } \ell \neq k. \]

(12)

The last inequality constraints ensure that different nodes on the same board have at least a minimal distance \( d_n \) from each other. Without this constraint for \( N > 1 \) the optimal solution would assign the same position for at least two different nodes on the same board. Note that the definition of the function \( T \) contains the integers \( m, n \) so that (12) is a quite difficult mixed integer optimization problem. Therefore, instead of this kind of optimal positioning we suggest to maximize the number of links that minimize the pathloss under all links with highest possible response. We call such a link SLHR (shortest link with highest response).

#### B. SLHR maximization

From now on we assume that \( M = 4 \). To illustrate the SLHR maximization problem we first focus on a special scenario and afterwards derive results for arbitrary numbers \( N \) of nodes on each board.

**Example 1.** We consider a board-to-board communication scenario where each board is equipped with \( N = 9 \) uniform quadratic antenna arrays of dimension \( 4 \times 4 \). Each antenna array uses the same beamforming network as described in Section III-B. Therefore, every node on each board can adjust \( M^2 = 16 \) spatial beams. The boards are parallel to the \( x-y \)-plane and have a distance of \( d \). For simplicity, we assume that one antenna array of the lower board is positioned at \( 0 = [0, 0, 0]^T \). Using (2) we can derive the MRAs for the 12 beams with highest possible response directly and can assign positions on the upper board lying on these axes. This is depicted in Fig. 4.

Each of these 12 positions on the upper board guarantees the highest possible response concerning the link to the node in \( 0 \). If a node is placed at a red circled position in Fig. 4 we obtain an SLHR between this node and the node at \( 0 \).

In order to position the nodes on each board we now aim at maximizing the total number of SLHRs among all links. As an initial guess, a symmetric positioning of the nodes seems reasonable and is shown in subfigure P1 of Fig. 5, where the positions of the nodes on the upper board are projected to the \( x-y \)-plane. The number of SLHRs for the positioning shown in subfigure P1 is 25. By moving two of the nodes we get a positioning with 27 SLHRs, see subfigure P2. In the following we show, that this number is indeed maximal.

In order to find an SLHR-optimal positioning of \( N \in \mathbb{N} \) nodes on each board we firstly derive some necessary conditions. Therefore the projection
\[ \pi_{xy} : \mathbb{R}^3 \to \mathbb{R}^2, (w_1, w_2, w_3)^T \mapsto (w_1, w_2)^T \]
of the two boards onto the \( x \)-\( y \)-plane is considered. Let \( h \) be half of the shortest distance between two projected nodes of the same board. Applying the transformation of coordinates

\[
T(\alpha, \tau) : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \tau \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}
\]

with \( \tau = \frac{1}{2d} \) and \( \alpha = \frac{\pi}{4} \), we can assume that w.l.o.g. the projected configuration lies on the (integer) grid lines of a standard Cartesian coordinate system. Consequently, each SLHR is situated on these grid lines as well. We call these a standard Cartesian coordinate system. Consequently, each projected configuration lies on the (integer) grid lines of \( R(N) \) is situated on these grid lines as well. We call these a standard Cartesian coordinate system. Consequently, each projected configuration lies on the (integer) grid lines of \( R(N) \).

Lemma 1. Let \( P(N) \) be an SLHR-optimal positioning in standard form for \( N \in \mathbb{N} \) nodes on each board and let \( R(N) \) denote a rectangle circumscribing \( P(N) \) with minimal perimeter. Then the following properties hold:

1) Each grid line \( G \) that divides \( P(N) \) in two nonempty subsets contains a point of \( P(N) \), i.e., \( P(N) \cap G \neq \emptyset \).
2) Each grid line \( G \) with \( R(N) \cap G \neq \emptyset \) contains a point of \( P(N) \), i.e., \( P(N) \cap G \neq \emptyset \).

Proof: Both assertions are proven indirectly:

1) If there were a grid line \( G \) with \( P(N) \cap G = \emptyset \) that divides \( P(N) \) in two nonempty subsets one could push together the two subsets such that additional SLHRs would be generated. Thus, \( P(N) \) cannot be optimal, which gives the contradiction.
2) Assuming that \( G \) were a grid line that violates assertion 2) the following two cases can occur. If \( G \) is a marginal grid line of \( R(N) \), the rectangle was not minimal with regard to the perimeter. Otherwise, if \( G \) is an interior grid line of \( R(N) \), it divides \( R(N) \) into two sub-rectangles. If one of them contains no points of \( P(N) \) the rectangle had no minimal perimeter. Elsewise, we have found a grid line partitioning \( P(N) \) that contains no points of \( P(N) \), which contradicts 1).

Now we are able to prove our first theorem:

Theorem 1. Let \( N \in \mathbb{N} \). Then,

\[
U(N) := 4N - \min \{a + b : a, b \in \mathbb{N}, a \cdot b \geq 2N\}
\]

is an upper bound for the maximal number of SLHRs.

Proof: We consider an SLHR-optimal positioning \( P(N) \) in standard form and an appropriate circumscribing rectangle \( R(N) \) with minimal perimeter. Since in the projected scenario each SLHR connects two nodes either vertically or horizontally, we can assign each SLHR to its left or upper node. Therefore, each of the \( 2N \) nodes can at most possess two assigned SLHRs leading to \( U_1(N) = 4N \) as an upper bound for their total number. Otherwise, there are also nodes that do not provide this maximal amount of two assignments. We now estimate their number:

With Lemma 1 we obtain that each (horizontal or vertical) grid line which intersects \( R(N) \) has to contain at least one point of \( P(N) \). Thus, each of these horizontal grid lines contains a farthest right point of \( P(N) \) and each of these vertical grid lines contains a farthest bottom point of \( P(N) \). These farthest right points cannot be left endpoints of an SLHR as well as farthest bottom points cannot be upper endpoints of an SLHR. Consequently, we can reduce \( U_1(N) \) by 1 for each of these marginal points. Let \( a, b \in \mathbb{N} \) denote the number of integer grid points lying on a horizontal and vertical rectangle side of \( R(N) \), respectively. Then, for each of these \( a + b \) grid points there is a corresponding horizontal or vertical grid line intersecting \( R(N) \), i.e., altogether \( U_1(N) \) has to be reduced by \( a + b \). Note that the perimeter

\[
p(N) = 2 \cdot [(a - 1) + (b - 1)] = 2(a + b) - 4
\]

of \( R(N) \) is minimal if and only if \( a + b \) is minimal. Thereby, we obtain our upper bound

\[
\#SLHR(N) \leq 4N - \min \{a + b : a, b \in \mathbb{N}, a \cdot b \geq 2N\} = U(N),
\]

where \#SLHR(N) denotes the maximal number of SLHRs. The constraint \( a \cdot b \geq 2N \) ensures that \( R(N) \) circumscribes \( P(N) \), since our \( 2N \) nodes have to be contained in the set of
integer grid points covered by \( R(N) \) whose cardinal number equals to the product \( a \cdot b \).

For simplicity, we define

\[
u(N) := \min \{a + b : a, b \in \mathbb{N}, a \cdot b \geq 2N \}.
\]

**Lemma 2.** For \( N \in \mathbb{N} \) the set

\[
S(N) := \{(c, d) \in \mathbb{N}^2 : c \cdot d \geq 2N, c + d = u(N)\}
\]

contains an element \((a, b) \in S(N)\) with \( a \leq b \) and \( b - a \leq 1 \).

**Proof:** Surely we have \( S(N) \neq \emptyset \) since the minimum \( u(N) \) exists for all \( N \in \mathbb{N} \). Additionally, note that \((c, d) \in S(N)\) implies \((d, c) \in S(N)\). Therefore, we can w.l.o.g. consider \((a, b) \in S(N)\) with \( a \leq b \). If \( b - a > 1 \), i.e. \( b - a \geq 2 \), we define

\[
(\tilde{a}, \tilde{b}) := (a + 1, b - 1).
\]

Obviously, we have \( \tilde{a}, \tilde{b} \in \mathbb{N} \) and \( \tilde{a} + \tilde{b} = u(N) \). Furthermore,

\[
\tilde{a} \cdot \tilde{b} = (a + 1) \cdot (b - 1) = ab + (b - a) - 1 \\
\geq ab + 2 - 1 > 2N
\]

holds. Consequently, we obtain \((\tilde{a}, \tilde{b}) \in S(N)\) and

\[
b - \tilde{a} = b - a - 2.
\]

If \( b - \tilde{a} \leq 1 \) is not satisfied yet we repeat the procedure and the assertion follows by mathematical induction.

The following theorem states that the upper bound presented in Theorem 1 is already optimal. Moreover, the proof contains an explicit positioning that provides this maximal number of SLHRs.

**Theorem 2.** For any \( N \in \mathbb{N} \) the upper bound \( U(N) \) is tight, i.e., \( \#SLHR(N) = U(N) \).

**Proof:** For \( N \in \mathbb{N} \) let us consider the positioning (counterclockwise spiral) \( P^*(N) \) in Fig. 6. We claim that the number \( \#SLHR_{P^*}(N) \) of SLHRs in this configuration is equal to \( U(N) \) and prove this by mathematical induction over \( N \in \mathbb{N} \).

1) **induction basis:**

For \( N = 1 \) positioning \( P^*(1) \) leads to one SLHR and on the other hand we have

\[
U(1) = 4 \cdot 1 = 4 \cdot 1 - (2 + 1) = 1.
\]

2) **induction hypothesis:**

The assertion is true for some \( N = k \in \mathbb{N} \), i.e., \( \#SLHR_{P^*}(k) = U(k) \).

3) **induction step:**

By means of Lemma 2 we can find \((a_k, b_k) \in S(k)\) with \( a_k \leq b_k \) and \( b_k - a_k \leq 1 \). Now, we have to consider two cases:

a) If \( a_k \cdot b_k \geq 2(k + 1) \) the rectangle \( R(k) \) still covers \( 2(k + 1) \) nodes and is minimal as well. Thus, we have \( R(k + 1) = R(k) \) and accordingly \( u(k + 1) = u(k) \).

The new positioning \( P^*(k + 1) \) contains 4 new SLHRs as exemplary depicted in Fig. 7. Indeed, we now have

\[
\#SLHR_{P^*}(k + 1) = \#SLHR_{P^*}(k) + 4
\]

\[
= U(k) + 4
\]

\[
= 4k - u(k) + 4
\]

\[
= 4(k + 1) - u(k)
\]

\[
= U(k + 1).
\]

b) If \( a_k \cdot b_k < 2(k + 1) \) we have to increase the rectangle, i.e. we take w.l.o.g. \( a_{k+1} = a_k + 1 \). Consequently, we have \( u(k + 1) = u(k) + 1 \).

Hence, the new optimal positioning \( P^*(k + 1) \) contains 3 new SLHRs as exemplary depicted in Fig. 8. In this case we have

\[
\#SLHR_{P^*}(k + 1) = \#SLHR_{P^*}(k) + 3
\]

\[
= U(k) + 3
\]

\[
= 4k - u(k) + 3
\]

\[
= 4(k + 1) - (u(k) + 1)
\]

\[
= 4(k + 1) - u(k + 1)
\]

\[
= U(k + 1).
\]

Since \( U(N) \) is an upper bound for \( \#SLHR(N) \) and the positioning \( P^*(N) \) leads to \( \#SLHR_{P^*}(N) = U(N) \) we obtain that the maximal number \( \#SLHR(N) \) is equal to \( U(N) \).

**Remark 2.** The configuration \( P^*(N) \) introduced in the proof of Theorem 2 is not necessarily the unique SLHR-optimal positioning. For example, the proposed positioning \( P_2 \) in Fig. 5 is slightly different to \( P^*(9) \) and also minimizes the overall pathloss (while maximizing the number of SLHRs).

**Remark 3.** A fact that we have not taken into account so far is the influence of interference. On the one hand, within the general setting it can be assumed that the noise caused by reflection is strongly damped and therefore can be neglected. On the other hand, the special structure of the Butler matrix
The distance between two neighboring nodes on the same board is 
\[ g = d \cdot \tan(\theta^* \cos(\varphi^*)) \]

The distance between two neighboring nodes on the same board is \( g = d \cdot \tan(\theta^* \cos(\varphi^*)) \). Now we can derive distances and array responses for all links. For the positioning \( P^* \) we obtain that nodes with the same distance also have the same array response. The following table shows the results for the positioning \( P^* \).

The trajectory of two neighboring nodes on the same board is \( g = d \cdot \tan(\theta^* \cos(\varphi^*)) \). Now we can derive distances and array responses for all links. For the positioning \( P^* \) we obtain that nodes with the same distance also have the same array response. The following table shows the results for the positioning \( P^* \).

From the fourth row of the Table I it can be seen that links with a relatively low array response can occur. A possibility to reduce this effect is to modify the geometry of the antenna arrays and/or the Butler matrix.

### Table I
#### ARRAY RESPONSES FOR THE POSITIONING \( P^* \)

<table>
<thead>
<tr>
<th>link distance</th>
<th>number of links</th>
<th>array response (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \sqrt{d^2 + 2g^2} ]</td>
<td>24</td>
<td>0</td>
</tr>
<tr>
<td>[ \sqrt{d^2 + 10g^2} ]</td>
<td>24</td>
<td>-1.16</td>
</tr>
<tr>
<td>[ \sqrt{d^2 + 18g^2} ]</td>
<td>8</td>
<td>-5.57</td>
</tr>
<tr>
<td>[ \sqrt{d^2 + 26g^2} ]</td>
<td>8</td>
<td>-0.55</td>
</tr>
<tr>
<td>total</td>
<td>64</td>
<td>-67.8</td>
</tr>
</tbody>
</table>

Since the minimization of the overall losses does not seem to be numerically tractable we suggested to maximize the number of shortest links with highest response (SLHR). In particular, we established an upper bound for the number of SLHRs depending on the number \( N \) of nodes on each board. Moreover, we were able to show, that this bound is tight and for every \( N \in \mathbb{N} \) we provided a concrete positioning \( P^*(N) \) that maximizes the number of SLHRs. Among all SLHR-maximal positionings there might be (small) differences in terms of overall pathloss. To enhance the array response of links which are not located at an MRA further optimization might be considered.

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**REFERENCES**